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THE $\mathcal{H}$-FLOW TRANSLATING SOLITONS IN $\mathbb{R}^3$ AND $\mathbb{R}^4$

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ABSTRACT. We solve the prescribed Hoffman-Osserman Gauss map problem for translating soliton surfaces to the mean curvature flow in $\mathbb{R}^4$. Our solution is inspired by Ilmanen's correspondence between translating soliton surfaces and minimal surfaces.

The recent decades admit intensive research devoted to the study of solitons [8] to the mean curvature flow ($\mathcal{H}$-flow for short). The simplest example is the grim reaper $y = \ln(\cos x)$ which moves by downward translation under the $\mathcal{H}$-flow. As known in [3, 9, 18, 21], there exist fascinating geometric dualities between the $\mathcal{H}$-flow solitons and minimal submanifolds.

A surface is a *translator* [21] when its mean curvature vector field agrees with the normal component of a constant Killing vector field. Translators arise as Hamilton's convex eternal solutions and Huisken-Sinestrari's Type II singularities for the $\mathcal{H}$-flow, and become natural generalization of minimal surfaces. The eight equivalent definitions of minimal surfaces illustrated in [12] show the richness of the minimal surfaces theory. However, even in $\mathbb{R}^3$, only few non-minimal translators are known:

Altschuler and Wu [1] showed the existence of the convex, rotationally symmetric, entire graphical translator. Clutterbuck, Schnürer and Schulze [4] constructed the winglike bigraphical translators, which are analogous to catenoids. Halldorsson [5] proved the existence of helicoidal translators. Nguyen [16] used Scherk's minimal towers to desingularize the intersection of a grim reaper cylinder and a plane, and obtained a complete embedded translator. See also her generalization [17].

Our main goal is to adopt the splitting of the generalized Gauss map of oriented surfaces in $\mathbb{R}^4$ to construct an explicit Weierstrass type representation for translators in $\mathbb{R}^4$. We first introduce the complexification of the generalized Gauss map. Inside the complex projective space $\mathbb{CP}^3$, we take the variety

$$\mathcal{Q}_2 = \{\, [\zeta] = [\zeta_1 : \cdots : \zeta_4] \in \mathbb{CP}^3 : {\zeta_1}^2 + \cdots + {\zeta_4}^2 = 0 \,\},$$

which becomes a model for the Grassmannian manifold $\mathcal{G}_{2,2}$ of oriented planes in $\mathbb{R}^4$. Reading the biholomorphic map from $\mathcal{Q}_2$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$ as a splitting of $\mathcal{Q}_2$, Hoffman and Osserman [6, 7] defined the generalized Gauss map of a conformal immersion $\mathbf{X} : \Sigma \to \mathbb{R}^4$, $z \mapsto \mathbf{X}(z)$ as follows:

$$\mathcal{G}(z) = \left[\frac{\partial \mathbf{X}}{\partial z} \right] = [\, 1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i\,(g_1 + g_2) \,] \in \mathcal{Q}_2 \subset \mathbb{CP}^3.$$

We call the induced pair (g_1, g_2) the complexified Gauss map of the immersion $\mathbf{X}$.

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Lemma 1 (Poincaré's Lemma). *Let $\xi : \Omega \to \mathbb{C}$ be a function on a simply connected domain $\Omega \subset \mathbb{C}$. If we have $\frac{\partial}{\partial \overline{z}}\xi(z) \in \mathbb{R}$ for all $z \in \Omega$, then there exists a function $x : \Omega \to \mathbb{R}$ such that $\frac{\partial}{\partial z}x(z) = \xi(z)$.*

Theorem 2 (Correspondence from null curves in $\mathbb{C}^4$ to translators in $\mathbb{R}^4$). *Let (g_1, g_2) be a pair of nowhere-holomorphic $\mathcal{C}^2$ functions from a simply connected domain $\Omega \subset \mathbb{C}$ to the open unit disc $\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}$ satisfying the compatibility condition*

$$\mathcal{F} := \frac{(g_1)_{\overline{z}}}{(1 - g_1\overline{g_2})\left(1 + |g_1|^2\right)} = \frac{(g_2)_{\overline{z}}}{(1 - \overline{g_1}g_2)\left(1 + |g_2|^2\right)}, \quad z \in \Omega. \tag{0.1}$$

We assume that one of the following two integrability conditions holds on Ω:

$$0 = (g_1)_{z\overline{z}} + \left(\frac{\overline{g_2}}{1 - g_1\overline{g_2}} - \frac{\overline{g_1}}{1 + |g_1|^2}\right)(g_1)_z (g_1)_{\overline{z}} + \frac{g_1 + g_2}{(1 - \overline{g_1}g_2)\left(1 + |g_1|^2\right)}|(g_1)_{\overline{z}}|^2, \tag{0.2}$$

$$0 = (g_2)_{z\overline{z}} + \left(\frac{\overline{g_1}}{1 - \overline{g_1}g_2} - \frac{\overline{g_2}}{1 + |g_2|^2}\right)(g_2)_z (g_2)_{\overline{z}} + \frac{g_1 + g_2}{(1 - g_1\overline{g_2})\left(1 + |g_2|^2\right)}|(g_2)_{\overline{z}}|^2. \tag{0.3}$$

Then, we obtain the following statements.
***(a)** Both (0.2) and (0.3) hold. (Assuming (0.1), we claim that (0.2) is equivalent to (0.3).)*
***(b)** The complex curve $\phi := (\phi_1, \phi_2, \phi_3, \phi_4) : \Omega \to \mathbb{C}^4$ defined by*

$$\phi = f\left(1 + g_1g_2, i(1 - g_1g_2), g_1 - g_2, -i(g_1 + g_2)\right), \quad f := -2i\overline{\mathcal{F}}$$

fulfills the three properties on the domain Ω:

- ***(b1) nullity*** $\phi \cdot \phi = {\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 + {\phi_4}^2 = 0$,
- ***(b2) non-degeneracy*** $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 > 0$,
- ***(b3) integrability*** $\frac{\partial \phi}{\partial \overline{z}} = \left(\frac{\partial \phi_1}{\partial \overline{z}}, \frac{\partial \phi_2}{\partial \overline{z}}, \frac{\partial \phi_3}{\partial \overline{z}}, \frac{\partial \phi_4}{\partial \overline{z}}\right) \in \mathbb{R}^4$.

***(c)** Integrating the complex null immersion $\phi : \Omega \to \mathbb{C}^4$ yields a translator Σ in $\mathbb{R}^4$.*

- ***(c1)** There exists a conformal immersion $\mathbf{X} = (x_1, x_2, x_3, x_4) : \Omega \to \mathbb{R}^4$ satisfying*

$$\mathbf{X}_z = \phi.$$

- ***(c2)** The induced metric ds^2 on the z-domain Ω by the immersion $\mathbf{X}$ reads*

$$ds^2 = \frac{16\,|(g_1)_{\overline{z}}|^2}{|1 - g_1\overline{g_2}|^2} \cdot \frac{1 + |g_2|^2}{1 + |g_1|^2}|dz|^2 = \frac{16\,|(g_2)_{\overline{z}}|^2}{|1 - \overline{g_1}g_2|^2} \cdot \frac{1 + |g_1|^2}{1 + |g_2|^2}|dz|^2.$$

- ***(c3)** The pair (g_1, g_2) is the complexified Gauss map of the surface $\Sigma = \mathbf{X}(\Omega)$. In other words, the generalized Gauss map of the conformal immersion $\mathbf{X}$ reads*

$$[\mathbf{X}_z] = [\,1 + g_1g_2, i(1 - g_1g_2), g_1 - g_2, -i\,(g_1 + g_2)\,] \in \mathcal{Q}_2 \subset \mathbb{CP}^3.$$

- ***(c4)** The surface Σ becomes a translator with the translating velocity $-\mathbf{e_4} = (0, 0, 0, -1)$.*

Proof. **Step A.** For the proof of **(a)**, we first set up the notations

$$\begin{cases} \mathcal{L} := (g_1)_{z\overline{z}} + \left(\frac{\overline{g_2}}{1-g_1\overline{g_2}} - \frac{\overline{g_1}}{1+|g_1|^2}\right)(g_1)_z (g_1)_{\overline{z}} + \frac{g_1+g_2}{(1-\overline{g_1}g_2)\left(1+|g_1|^2\right)}|(g_1)_{\overline{z}}|^2, \\ \mathcal{R} := (g_2)_{z\overline{z}} + \left(\frac{\overline{g_1}}{1-\overline{g_1}g_2} - \frac{\overline{g_2}}{1+|g_2|^2}\right)(g_2)_z (g_2)_{\overline{z}} + \frac{g_1+g_2}{(1-g_1\overline{g_2})\left(1+|g_2|^2\right)}|(g_2)_{\overline{z}}|^2. \end{cases}$$

We first assume only (0.1). Taking the conjugation in (0.1) yields

$$\overline{\mathcal{F}} = \frac{(\overline{g_2})_z}{(1-g_1\overline{g_2})\left(1+|g_2|^2\right)} = \frac{(\overline{g_1})_z}{(1-\overline{g_1}g_2))\left(1+|g_1|^2\right)}.$$

Taking into account this, we deduce

$$\begin{aligned}\frac{\mathcal{F}_z}{\mathcal{F}} &= \frac{(g_1)_{z\overline{z}}}{(g_1)_{\overline{z}}} + \left(\frac{\overline{g_2}}{1-g_1\overline{g_2}} - \frac{\overline{g_1}}{1+|g_1|^2}\right)(g_1)_z + \left(\frac{1+|g_2|^2}{1-\overline{g_1}g_2} - 1\right)\cdot\frac{g_1}{1+|g_1|^2}(\overline{g_1})_z \\ &= \frac{\mathcal{L}}{(g_1)_{\overline{z}}} - \overline{F}\left[g_1\left(1-|g_2|^2\right) + g_2\left(1-|g_1|^2\right)\right]\end{aligned}$$

and

$$\begin{aligned}\frac{\mathcal{F}_z}{\mathcal{F}} &= \frac{(g_2)_{z\overline{z}}}{(g_2)_{\overline{z}}} + \left(\frac{\overline{g_1}}{1-g_2\overline{g_2}} - \frac{\overline{g_2}}{1+|g_2|^2}\right)(g_2)_z + \left(\frac{1+|g_1|^2}{1-g_1\overline{g_2}} - 1\right)\cdot\frac{g_2}{1+|g_2|^2}(\overline{g_2})_z \\ &= \frac{\mathcal{R}}{(g_2)_{\overline{z}}} - \overline{F}\left[g_1\left(1-|g_2|^2\right) + g_2\left(1-|g_1|^2\right)\right].\end{aligned}$$

These two equalities thus show the equality

$$\frac{\mathcal{L}}{(g_1)_{\overline{z}}} = \frac{\mathcal{R}}{(g_2)_{\overline{z}}},$$

which means the desired implications: (0.2) $\Longleftrightarrow \mathcal{L} = 0 \Longleftrightarrow \mathcal{R} = 0 \Longleftrightarrow$ (0.3).

Step B. We deduce several equalities which will be used in the proof of **(b)** and **(c)**. According to **(a)**, from now on, we assume that both (0.2) and (0.3) hold. Since both $\mathcal{L}$ and $\mathcal{R}$ vanish, the previous equalities imply

$$\mathcal{F}_z = -|\mathcal{F}|^2\left[g_1\left(1-|g_2|^2\right) + g_2\left(1-|g_1|^2\right)\right].$$

Conjugating this and using the definition $f = -2i\overline{\mathcal{F}}$, we arrive at the equality

$$f_{\overline{z}} = \frac{i}{2}|f|^2\left[\overline{g_1}\left(1-|g_2|^2\right) + \overline{g_2}\left(1-|g_1|^2\right)\right]. \tag{0.4}$$

The compatibility condition (0.1) can be written in terms of $f = -2i\overline{\mathcal{F}}$:

$$\overline{f} = \frac{2i(g_1)_{\overline{z}}}{(1-g_1\overline{g_2})\left(1+|g_1|^2\right)} = \frac{2i(g_2)_{\overline{z}}}{(1-\overline{g_1}g_2)\left(1+|g_2|^2\right)}. \tag{0.5}$$

It immediately follows from (0.4) and (0.5) that

$$(fg_1)_{\overline{z}} = f_{\overline{z}}g_1 + (g_1)_{\overline{z}}f = -\frac{i}{2}|f|^2\left(1-2g_1\overline{g_2} + |g_1|^2|g_2|^2\right) \tag{0.6}$$

and

$$(fg_2)_{\overline{z}} = f_{\overline{z}}g_2 + (g_2)_{\overline{z}}f = -\frac{i}{2}|f|^2\left(1-2\overline{g_1}g_2 + |g_1|^2|g_2|^2\right). \tag{0.7}$$

Another computation taking into account (0.5) and (0.6) shows

$$(fg_1g_2)_{\overline{z}} = (fg_1)_{\overline{z}}g_2 + (g_2)_{\overline{z}}fg_1 = -\frac{i}{2}|f|^2\left[g_1\left(1-|g_2|^2\right) + g_2\left(1-|g_1|^2\right)\right]. \tag{0.8}$$

Step C. Our aim here is to establish the claims in **(b)** on the complex curve

$$\phi = (\phi_1,\phi_2,\phi_3,\phi_4) = f\,(\,1+g_1g_2, i(1-g_1g_2), g_1-g_2, -i(g_1+g_2)\,).$$

First, the equality in **(b1)** is obvious. Next, by the assumptions on g_1 and g_2, we see that $f = -2i\overline{\mathcal{F}}$ never vanish. Then, the assertion **(b2)** follows from the equality

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 = 2|f|^2 \left(1 + |g_1|^2\right)\left(1 + |g_2|^2\right). \tag{0.9}$$

We employ the equalities in **Step B** to show the assertion **(b3)**. Joining the equalities in (0.4), (0.6), (0.7), and (0.8) and the definition of ϕ, we reach

$$\begin{cases} (\phi_1)_{\overline{z}} = \ \ |f|^2 \left[\left(1 - |g_2|^2\right)\operatorname{Im} g_1 + \left(1 - |g_1|^2\right)\operatorname{Im} g_2\right], \\ (\phi_2)_{\overline{z}} = -|f|^2 \left[\left(1 - |g_2|^2\right)\operatorname{Re} g_1 + \left(1 - |g_1|^2\right)\operatorname{Re} g_2\right], \\ (\phi_3)_{\overline{z}} = \ 2|f|^2 \ \mathrm{Im}\ (\overline{g_1} g_2), \\ (\phi_4)_{\overline{z}} = -|f|^2 \left[1 - 2\mathrm{Re}\ (\overline{g_1} g_2) + |g_1|^2 |g_2|^2\right]. \end{cases} \tag{0.10}$$

These four equalities guarantee the integrability condition $\left(\frac{\partial \phi_1}{\partial \overline{z}}, \frac{\partial \phi_2}{\partial \overline{z}}, \frac{\partial \phi_3}{\partial \overline{z}}, \frac{\partial \phi_4}{\partial \overline{z}}\right) \in \mathbb{R}^4$.

Step D. We prove the claims **(c1)**, **(c2)**, and **(c3)**. Thanks to **(b3)**, we can integrate the curve ϕ. Since Ω is simply connected, applying Lemma 1 to the complex curve ϕ, we see the existence of the function $\mathbf{X} = (x_1, x_2, x_3, x_4) : \Omega \to \mathbb{R}^4$ satisfying

$$\mathbf{X}_z = \phi = f\,(\,1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)\,)\,.$$

This and the nullity of ϕ guarantee that the mapping $\mathbf{X}$ is conformal. Using (0.9), one then find that the induced metric $ds^2 = \Lambda^2 |dz|^2$ by the immersion $\mathbf{X}$ reads

$$ds^2 = \Lambda^2 |dz|^2 = 4|f|^2 \left(1 + |g_1|^2\right)\left(1 + |g_2|^2\right)|dz|^2. \tag{0.11}$$

Since f never vanish, this completes the proof of **(c1)**. Also, joining (0.5) and (0.11) imply the equality in **(c2)**. The integrability $\mathbf{X}_z = \phi$ and the definition of ϕ gives

$$[\mathbf{X}_z] = [1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i\,(g_1 + g_2)]\,,$$

which completes the proof of **(c3)**.

Step E. Finally, we prove the claim **(c4)**. First, we find the normal component of the vector field $-\mathbf{e_4} = (0, 0, 0, -1)$ in terms of g_1 and g_2. We compute

$$\begin{aligned} (-\mathbf{e_4})^\perp &= -\mathbf{e_4} - \left[\left(\frac{\mathbf{X}_u}{\Lambda} \cdot (-\mathbf{e_4})\right)\frac{\mathbf{X}_u}{\Lambda} + \left(\frac{\mathbf{X}_v}{\Lambda} \cdot (-\mathbf{e_4})\right)\frac{\mathbf{X}_v}{\Lambda}\right] \\ &= -\mathbf{e_4} + \frac{2}{\Lambda^2}\left[\left(\mathbf{X}_{\overline{z}} \cdot \mathbf{e_4}\right)\mathbf{X}_z + \left(\mathbf{X}_z \cdot \mathbf{e_4}\right)\mathbf{X}_{\overline{z}}\right] \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \frac{4}{\Lambda^2}\begin{bmatrix} \mathrm{Re}\ \left(\phi_1 \overline{\phi_4}\right) \\ \mathrm{Re}\ \left(\phi_2 \overline{\phi_4}\right) \\ \mathrm{Re}\ \left(\phi_3 \overline{\phi_4}\right) \\ |\phi_4|^2. \end{bmatrix}. \end{aligned}$$

Combining this, (0.10), and (0.11) yields

$$(-\mathbf{e_4})^\perp = \frac{1}{\left(1 + |g_1|^2\right)\left(1 + |g_2|^2\right)}\begin{bmatrix} \left[\left(1 - |g_2|^2\right)\operatorname{Im} g_1 + \left(1 - |g_1|^2\right)\operatorname{Im} g_2\right] \\ -\left[\left(1 - |g_2|^2\right)\operatorname{Re} g_1 + \left(1 - |g_1|^2\right)\operatorname{Re} g_2\right] \\ 2\ \mathrm{Im}\ (\overline{g_1} g_2) \\ -\left[1 - 2\mathrm{Re}\ (\overline{g_1} g_2) + |g_1|^2 |g_2|^2\right] \end{bmatrix}.$$

Second, we find the mean curvature vector $\mathcal{H} = \triangle_{ds^2}\mathbf{X} = \frac{4}{\Lambda^2}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial z}\mathbf{X}\right) = \frac{4}{\Lambda^2}\phi_{\overline{z}}$ on our surface $\Sigma = \mathbf{X}(\Omega)$. Now, joining this, (0.10), and (0.11), we can write the mean curvature vector $\mathcal{H}$ in terms of g_1 and g_2:

$$\mathcal{H} = \frac{1}{\left(1+|g_1|^2\right)\left(1+|g_2|^2\right)} \begin{bmatrix} \left[\left(1-|g_2|^2\right)\operatorname{Im} g_1 + \left(1-|g_1|^2\right)\operatorname{Im} g_2\right] \\ -\left[\left(1-|g_2|^2\right)\operatorname{Re} g_1 + \left(1-|g_1|^2\right)\operatorname{Re} g_2\right] \\ 2\operatorname{Im}\left(\overline{g_1}g_2\right) \\ -\left[1-2\operatorname{Re}\left(\overline{g_1}g_2\right)+|g_1|^2|g_2|^2\right] \end{bmatrix}.$$

We therefore conclude that $\mathcal{H} = (-\mathbf{e_4})^{\perp}$. $\square$

Remark 1 (**Ilmanen's correspondence**). Theorem 2 generalizes the classical Weierstrass construction from holomorphic null immersions in $\mathbb{C}^3$ to conformal minimal immersions in $\mathbb{R}^3$. The key ingredient behind Theorem 2 is the Ilmanen correspondence between translators and minimal surfaces. (See [9] and [21].) We deform the flat metric of $\mathbb{R}^4$ conformally to introduce the four dimensional Riemannian manifold

$$\mathcal{I}^4 = \left(\mathbb{R}^4,\, e^{-x_4}\left({dx_1}^2+{dx_2}^2+{dx_3}^2+{dx_4}^2\right)\right).$$

Any conformal immersion $\mathbf{X}:\Omega\to\mathbb{R}^4$ of a downward translator with the translating velocity $-\mathbf{e_4}=(0,0,0,-1)$ in Euclidean space $\mathbb{R}^4$ then can be identified as a conformal minimal immersion $\mathbf{X}:\Omega\to\mathcal{I}^4$. However, it is not easy to find Riemannian manifolds which admit explicit representations for their minimal surfaces.

Example 3 (**The Hamiltonian stationary Lagrangian translator in** $\mathbb{C}^2$)**.** Recently, interesting Lagrangian translators in the complex plane $\mathbb{C}^2$ are discovered in [2, 10, 13]. In 2010, Castro and Lerma [2, Corollary 2] classified all Hamiltonian stationary Lagrangian translators in $\mathbb{C}^2$. Locally, they are unique up to dilations (except for the totally geodesic ones) [2, Corollary 3]. The point of this example is to explicitly recover the Hoffman-Osserman Gauss map of the Castro-Lerma translator in $\mathbb{R}^4=\mathbb{C}^2$.

We first notice that Theorem 2 still holds when we regard the prescribed Gauss map (g_1,g_2) as a pair of functions from a simply connected domain Ω to the complex plane (not just the unit disc). However, in this case, the induced mapping $\mathbf{X}:\Omega\to\mathbb{R}^4$ of the translator may admit the branch points where $\overline{g_1}g_2=1$ (or equivalently, $g_1\overline{g_2}=1$).

Imposing the additional condition $|g_1|=1$ produces Lagrangian translators with the velocity $-\mathbf{e_4}=(0,0,0,-1)$. Then, our integrability condition in **(c1)** for downward translators can be re-written as

$$\mathbf{X}_z = \left(\,(x_1)_z,(x_2)_z,(x_3)_z,(x_4)_z\,\right) = -\theta_z\left(\frac{1+g_1g_2}{g_1-g_2},\, i\frac{1-g_1g_2}{g_1-g_2},\, 1,\, -i\frac{g_1+g_2}{g_1-g_2}\right),$$

where θ denotes the Lagrangian angle with $ig_1=e^{i\theta}$. The third terms $(x_3)_z=-\theta_z$ can be viewed as [2, Proposition 1], [10, Proposition 2.5] and [15, Proposition 2.1].

We consider a complexified Gauss map of the form, for some $\mathbb{R}$-valued function $\mathcal{G}$,

$$\left(\,g_1(z),\, g_2(z)\,\right) = \left(\,e^{iv},\, \mathcal{G}(u)e^{iv}\,\right), \quad z=u+iv\in\mathbb{R}+i\mathbb{R}$$

and want to solve the system (0.1) and (0.2). First, the compatibility condition (0.1) induces the ordinary differential equation

$$\frac{1}{2} = \frac{1}{1+\mathcal{G}^2}\left(\mathcal{G}-\frac{d\mathcal{G}}{du}\right),$$

and a canonical solution is given by $\mathcal{G}(u) = \frac{u+1}{u-1}$. One can easily check that

$$(g_1(z),\, g_2(z)) = \left(e^{iv},\, \mathcal{G}(u)e^{iv} \right) = \left(e^{iv},\, \frac{u+1}{u-1}e^{iv} \right)$$

satisfies the integrability condition (0.2). Then, the induced Lagrangian translator Σ with the velocity $-\mathbf{e_4}$ admits the conformal parametrization

$$\mathbf{X}(u,v) = \left(u\sin v, -u\cos v, -v, -\frac{1}{2}u^2 \right).$$

Since the induced metric on Σ reads $ds^2 = \left(1+u^2\right)\left(du^2+dv^2\right)$, the Lagrangian angle function $\theta(u,v) = \frac{\pi}{2} + v$ with $ig_1 = e^{i\theta}$ is harmonic on Σ. We find that this Hamiltonian stationary Lagrangian translator Σ with the velocity $(0,0,0,-1)$ coincides with the Castro-Lerma translator [2, Corollary 2] with the velocity $(1,0,0,0)$ by a suitable change of coordinates.

Theorem 4 (Correspondence from null curves in $\mathbb{C}^3$ to translators in $\mathbb{R}^3$). *When a nowhere-holomorphic $\mathcal{C}^2$ function $G : \Omega \to \mathbb{D}$ from a simply connected domain $\Omega \subset \mathbb{C}$ to the open unit disc $\mathbb{D} := \{w \in \mathbb{C} \mid |w| < 1\}$ satisfying the translator equation*

$$G_{z\overline{z}} + 2\frac{\overline{G}\,|G|^2}{1-|G|^4}G_z G_{\overline{z}} + 2\frac{G}{1-|G|^4}|G_{\overline{z}}|^2 = 0, \quad z \in \Omega, \tag{0.12}$$

we associate a complex curve $\phi = \phi_{_G} = (\phi_1, \phi_2, \phi_3) : \Omega \to \mathbb{C}^3$ as follows:

$$\phi = \frac{2\overline{G}_z}{|G|^4 - 1}\left(1 - G^2, i\left(1+G^2\right), 2G\right).$$

(a) *Then, the complex curve ϕ fulfills the three properties on the domain Ω:*

- ***(a1) nullity*** $\phi \cdot \phi = {\phi_1}^2 + {\phi_2}^2 + {\phi_3}^2 = 0$,
- ***(a2) non-degeneracy*** $|\phi|^2 = |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0$,
- ***(a3) integrability*** $\frac{\partial \phi}{\partial \overline{z}} = \left(\frac{\partial \phi_1}{\partial \overline{z}}, \frac{\partial \phi_2}{\partial \overline{z}}, \frac{\partial \phi_3}{\partial \overline{z}} \right) \in \mathbb{R}^3$.

(b) *Also, integrating $\mathbf{X}_z = \phi$ on Ω yields a downward translator $\Sigma = \mathbf{X}(\Omega)$ with the velocity $-\mathbf{e_3} = (0,0,-1)$ in $\mathbb{R}^3$. The prescribed map G becomes the complexified Gauss map of the induced surface $\Sigma = \mathbf{X}(\Omega)$ via the stereographic projection from the north pole. The induced metric ds^2 by the immersion $\mathbf{X}$ reads $ds^2 = \frac{16\,|G_{\overline{z}}|^2}{\left(|G|^2-1\right)^2}|dz|^2$.*

Proof. We take $(g_1, g_2) = (iG, iG)$ in Theorem 2. □

Example 5 (Downward grim reaper cylinder as an analogue of Scherk's surface).
(a) An application of our representation formula in Theorem 4 to the solution

$$G(z) = G(u+iv) = \tanh u \in (-1,1), \quad u+iv \in \mathbb{C}$$

of the translator equation (0.12) yields the conformal immersion $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$

$$\mathbf{X}(u,v) = (x_1, x_2, x_3) = \left(-2\tan^{-1}(\tanh u),\, 2v,\, -\ln(\cosh(2u))\right),$$

(b) It represents the graphical translator with the translating velocity $-\mathbf{e_3}$:

$$x_3 = \mathcal{F}(x_1, x_2) = \ln(\cos x_1), \quad (x_1, x_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}.$$

Its height function $\mathcal{F} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \to \mathbb{R}$ is a Jenkins-Serrin type solution of

$$\nabla \cdot \left(\frac{1}{\sqrt{1 + |\nabla \mathcal{F}|^2}} \nabla \mathcal{F} \right) + \frac{1}{\sqrt{1 + |\nabla \mathcal{F}|^2}} = 0 \tag{0.13}$$

and has $-\infty$ boundary values. Our graph $x_3 = \mathcal{F}(x_1, x_2)$ becomes a cylinder over the downward grim reaper on the x_1x_3-plane. It can be viewed as an analogue of the classical Jenkins-Serrin type minimal graph, discovered by Scherk in 1834,

$$x_3 = \ln(\cos x_1) - \ln(\cos x_2), \quad (x_1, x_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Its height function takes the values $\pm\infty$ on alternate sides of the square domain.
(c) More generally, Scherk discovered the doubly periodic minimal graph $\Sigma_\rho^{2\alpha}$ [14]:

$$x_3 = \frac{1}{\rho} \left[\ln \left(\cos \left(\frac{\rho}{2} \left[\frac{x_1}{\cos \alpha} - \frac{x_2}{\sin \alpha} \right] \right) \right) - \ln \left(\cos \left(\frac{\rho}{2} \left[\frac{x_1}{\cos \alpha} + \frac{x_2}{\sin \alpha} \right] \right) \right) \right]$$

for some constants $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\rho > 0$. It is defined on an infinite chess board-like net of *rhomboids*. Its picture is available at [20]. However, unlike the deformations of Scherk's minimal surfaces by *shearing*, it is not possible to shear the grim reaper cylinder to obtain non-trivial deformations of unit-speed translators.

Remark 2 (**Jenkins-Serrin type problem for graphical translators**). A beautiful theory for infinite boundary value problems of minimal graphs is developed by Jenkins and Serrin [11]. Moreover, Spruck [19] obtained a Jenkins-Serrin type theory for constant mean curvature graphs. It would be very interesting to investigate a similar Dirichlet problem for graphical translators. In the following Example 6, we prove that, for any $l \geq \pi$, there exists a downward *unit-speed* graphical translator that its height function is defined over an infinite strip of width l and takes the values $-\infty$ on its boundary. We propose a conjecture that the lower bound π is a critical constant in the sense that, for any $l \in (0, \pi)$, there exists no downward *unit-speed* graphical translator defined over an infinite strip of width l approaching $-\infty$ on its boundary.

Example 6 (**Deformations of grim reaper cylinder).** Let $\theta \in \mathbb{R}$ be a constant.
(a) We begin with the following solution $G = G^\theta(z)$ of the translator equation (0.12):

$$G(z) = G(u + iv) = \frac{\cosh\theta \sinh(2u) + i \sinh\theta}{1 + \cosh\theta \cosh(2u)}, \quad u + iv \in \mathbb{C}.$$

Theorem 4 then induces the conformal immersion $\mathbf{X}^\theta = (x_1, x_2, x_3) : \mathbb{R}^2 \to \mathbb{R}^3$

$$\begin{cases} x_1(u, v) = -2\cosh\theta \tan^{-1}(\tanh u), \\ x_2(u, v) = \sinh\theta \ln(\cosh(2u)) + 2v, \\ x_3(u, v) = -\ln(\cosh(2u)) + 2v \sinh\theta. \end{cases}$$

The downward translator $\mathbb{G}^\theta = \mathbf{X}^\theta\left(\mathbb{R}^2\right)$ has the translating velocity $(0, 0, -1)$.
(b) Using the patch $\mathbf{X}^\theta$, one can easily check that Gauss map of the translator $\mathbb{G}^\theta$ lies on a half circle. Let us introduce a new linear coordinate

$$x_0 = \frac{1}{\cosh\theta} x_2 + \frac{\sinh\theta}{\cosh\theta} x_3$$

and then prepare an orthonormal basis

$$\mathcal{U}_1 = (1,0,0)\,,\ {\mathcal{U}_2}^{\theta} = \left(0, -\frac{\sinh\theta}{\cosh\theta}, \frac{1}{\cosh\theta}\right)\,,\ {\mathcal{U}_3}^{\theta} = \left(0, \frac{1}{\cosh\theta}, \frac{\sinh\theta}{\cosh\theta}\right).$$

It is easily shown that our surface $\mathbb{G}^{\theta}$ admits a new geometric patch

$$(x_1, x_2, x_3) = \widehat{\mathbf{X}}^{\theta}\,(x_1, x_0) = x_1\,\mathcal{U}_1 + \mathbf{T}^{\theta}(x_1)\,{\mathcal{U}_2}^{\theta} + x_0\,{\mathcal{U}_3}^{\theta}.$$

Here, $\mathbf{T}^{\theta}(\cdot) = \cosh\theta\ln\left(\cos\left(\frac{\cdot}{\cosh\theta}\right)\right)$ is a parabolic rescaling of the downward unit-speed grim reaper function. The patch $\widehat{\mathbf{X}}^{\theta}$ says that the surface $\mathbb{G}^{\theta}$ becomes a cylinder over a parabolically rescaled grim reaper curve in the plane spanned by $\mathcal{U}_1$ and ${\mathcal{U}_2}^{\theta}$.

(c) Our one-parameter family $\{\mathbb{G}^{\theta}\}_{\theta\in\mathbb{R}}$ of cylinders with the same translating velocity admits a simple geometric description. Applying a suitable rotation in the ambient space $\mathbb{R}^3$ to the grim reaper cylinder $\mathbb{G}^0$ with velocity $-{\mathcal{U}_2}^0 = (0,0,-1)$:

$$(x_1, x_0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \ \mapsto\ \widehat{\mathbf{X}}^{0}\,(x_1, x_0) = x_1\,\mathcal{U}_1 + \mathbf{T}^{0}(x_1)\,{\mathcal{U}_2}^{0} + x_0\,{\mathcal{U}_3}^{0},$$

we obtain the congruent cylinder parametrized by

$$(x_1, x_0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \ \mapsto\ x_1\,\mathcal{U}_1 + \mathbf{T}^{0}(x_1)\,{\mathcal{U}_2}^{\theta} + x_0\,{\mathcal{U}_3}^{\theta},$$

which translates with the rotated velocity $-{\mathcal{U}_2}^{\theta}$ under the $\mathcal{H}$-flow. However, we observe that this rotated cylinder also can be viewed as a translator with new velocity $-\cosh\theta\,{\mathcal{U}_2}^0 = (0,0,-\cosh\theta)$. Employing the appropriate parabolic rescaling as a speed-down action, we meet our cylinder $\mathbb{G}^{\theta}$ parametrized by

$$(x_1, x_0) \in \left(-\frac{\pi}{2}\cosh\theta, \frac{\pi}{2}\cosh\theta\right) \times \mathbb{R} \ \mapsto\ \widehat{\mathbf{X}}^{\theta}\,(x_1, x_0) = x_1\,\mathcal{U}_1 + \mathbf{T}^{\theta}(x_1)\,{\mathcal{U}_2}^{\theta} + x_0\,{\mathcal{U}_3}^{\theta},$$

which translates with velocity $-{\mathcal{U}_2}^0 = (0,0,-1)$ under the $\mathcal{H}$-flow.

(d) We prove the claim in Remark 2. We are able to view the downward unit-speed translator $\mathbb{G}^{\theta}$ as the graph of the function $\mathcal{F}^{\theta} : \left(-\frac{\pi}{2}\cosh\theta, \frac{\pi}{2}\cosh\theta\right) \times \mathbb{R} \to \mathbb{R}$:

$$x_3 = \mathcal{F}^{\theta}\,(x_1, x_2) = \cosh\theta\,\mathbf{T}^{\theta}(x_1) + \sinh\theta\ x_2.$$

Its height function $\mathcal{F}^{\theta}$ solves the PDE (0.13) over the strip of width $l = \pi\cosh\theta \geq \pi$.

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